

# Consistency-driven methodology to manage incomplete linguistic preference relation: A perspective based on personalized individual semantics

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**Abstract**—In linguistic decision making problems there may be cases when decision makers will not be able to provide complete linguistic preference relations. However, when estimating unknown linguistic preference values in incomplete preference relations, the existing research approaches ignore the fact that words mean different things for different people, i.e. decision makers have personalized individual semantics (PISs) regarding words. To manage incomplete linguistic preference relations with PISs, in this paper we propose a consistency-driven methodology both to estimate the incomplete linguistic preference values and to obtain the personalized numerical meanings of linguistic values of the different decision makers. The proposed incomplete linguistic preference estimation method combines the characteristic of the personalized representation of decision makers and guarantees the optimum consistency of incomplete linguistic preference relations in the implementation process. Numerical examples and a comparative analysis are included to justify the feasibility of the PISs based incomplete linguistic preference estimation method.

**Index Terms**—Personalized individual semantics, incomplete linguistic preference relation, consistency, linguistic decision making

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## I. INTRODUCTION

In real group decision making (GDM) activity, decision makers may prefer to use linguistic information instead of numerical numbers to represent their preferences. In this case, we deal with what is called a linguistic group decision making (LGDM). The LGDM problem aims to finding the best alternative(s) from a set of potential alternatives based on the linguistic preferences expressed by a group of decision makers [2], [11]. Particularly, linguistic preference relations [13], [40] are commonly used in LGDM to express decision makers' preferences over alternatives.

A difficulty in dealing with preference relations in GDM problems is the missing of some of the expected preference information values [10], [15], [22], [32]. Experts may not provide all the expected preference degree between two or more alternatives because the number of alternatives is high or because lack of knowledge on some of the pairwise comparisons. In such situation, experts provide incomplete preference relations. Generally, two types of methods have been proposed to manage incomplete preference relations based on consistency measurements:

(i) The iterative procedure, which estimates missing values between two alternatives based on all possible indirect paths between such alternatives, using intermediate alternatives, for which preference values are known.

(ii) The optimization-based procedure that obtains a complete linguistic preference relation with optimum consistency.

The iterative procedure and optimization-based procedure are both extensively investigated in various types of preference relations, such as additive preference relation (or fuzzy preference relation) [9], [27], [51], multiplicative preference relation [5], [28], interval-valued preference relation [34], [35], [44]. These approaches are also reported to manage incomplete linguistic preference relations [1], [3], [45] through linguistic additive consistency measurement [4], [7], [41], [42]. Based on the concept of additive consistency, Alonso et al. [1] proposed the iterative procedure to obtain the complete linguistic preference relations, which only uses the available values provided by the experts. Cabrerizo et al. [3] developed the iterative procedure for LGDM under an incomplete unbalanced linguistic environment. Zhao et al. [45] proposed the optimization-based model to estimate the missing values in

incomplete linguistic preference relations based on consistency measurement. The state-of-the-art survey about the management of incomplete preference relations has been presented in [21], [32].

It is argued and accepted that words mean different things for different people [25], [26] and, therefore, in LGDM decision makers have personalized individual semantics (PISs) [18] regarding words. For example, if two researchers reviewing an article think that the reviewed article is interesting, it may be the case that the word “interesting” would have different numerical interpretations or meanings to them. If the two researchers are asked to do semantics modelling of the word “interesting”, the numerical meaning of word “interesting” may be 0.9 for one researcher and 0.7 for the other researcher. The difference in the numerical meanings shows the PISs for researchers. It is even recognized that the concept of type-1 fuzzy set is not sufficient to represent the multiple meanings of words and, therefore, the concept of type-2 fuzzy set was proposed [25]. However, the type-2 fuzzy set concept does not capture in its definition the specific decision makers’ semantics. It is noted that to represent the PISs, the consistency-driven optimization model to personalize individual semantics of words for decision makers has been initiated in [18]. Based on the PIS model in [18], Huang et al. [17] proposed a new consensus reaching process in LGDM. Li et al. [19], [20], Zhang et al. [48] and Tang et al. [30], [31] studied the consistency-driven approaches in hesitant LGDM, large-scale LGDM, and distribution linguistic GDM, respectively, to show the PISs. The PIS model has also been applied in failure modes and effects analysis [49] and opinion dynamics [23].

An interesting, and worth to investigate, issue is that of estimating the missing values of the incomplete linguistic preference relations by considering the different individual semantics of decision makers. However, although the existing studies provide various methods to manage incomplete linguistic preference relations, they do not consider the decision makers’ PISs. Therefore, in this paper we propose a two-phase consistency-driven methodology to manage incomplete linguistic preference relations with PISs:

(1) In the first phase, a PIS based consistency-driven optimization (PIS-CDO) model is proposed to find out the set of possible personalized numerical scales (PNSs) that guarantee the optimum consistency of incomplete preference relations, which constitutes a foundational constraint condition to manage incomplete linguistic preference relations.

(2) In the second phase, the incomplete preference estimation based consistency-driven optimization (IPE-CDO) model to estimate the missing values in incomplete linguistic preference relations with PISs is developed.

Finally, we further illustrate the use and explain the feasibility of the incomplete preference estimation method with PISs. Numerical examples for the proposed method are provided and a comparative study with the existing methods, which does not implement PISs, is carried out. The main features of the incomplete preference estimation method with PISs are that it integrates the characteristic of the personalized representation of linguistic preferences, and guarantees the

optimum consistency of incomplete linguistic preference relations.

The rest of this paper is arranged as follows. In Section II, we present some basic preliminaries regarding LGDM that is necessary to develop our proposal. Then, in Section III the PIS-CDO model and the IPE-CDO model are both proposed to manage incomplete linguistic preference relations with PISs. Next, Section IV provides numerical examples to illustrate our proposal while Section V includes details of a comparative study with the existing studies that do not implement PISs by numerical analysis. Finally, Section VI concludes this paper.

## II. PRELIMINARIES

In this section, we introduce the basic knowledge regarding the 2-tuple linguistic model, linguistic preference relations, and the numerical scale model based on PIS, which is necessary to develop our proposal.

### A. The 2-tuple linguistic model

Let  $S = \{s_i | i = 0, 1, \dots, g\}$  be a linguistic term set. The linguistic term  $s_i$  represents a possible value of a linguistic variable [24], [46]. Herrera and Martínez [14] proposed the below 2-tuple linguistic model for computing with words.

**Definition 1** [14]. Let  $S$  be defined as above, let  $\beta \in [0, g]$  be a value representing the result of a symbolic aggregation operation. The 2-tuple linguistic model defines the transformation functions between 2-tuples and numerical values:

$$\Delta: [0, g] \rightarrow \bar{S} \quad (1)$$

being

$$\Delta(\beta) = (s_i, \alpha), \quad (2)$$

$$\text{with } \begin{cases} s_i, & i = \text{round}(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5] \end{cases}$$

The inverse function of  $\Delta$ ,  $\Delta^{-1}: \bar{S} \rightarrow [0, g]$  is defined as  $\Delta^{-1}(s_i, \alpha) = i + \alpha$ .

The computational model for linguistic 2-tuples [14] is the following:

(1) Comparison operator for linguistic 2-tuples: Let  $(s_k, \alpha)$  and  $(s_l, \gamma)$  be any two 2-tuples, then

(i) if  $k > l$ , then  $(s_k, \alpha)$  is larger than  $(s_l, \gamma)$ .

(ii) if  $k = l$ , then

(a) if  $\alpha = \gamma$ , then  $(s_k, \alpha)$ ,  $(s_l, \gamma)$  represents the same information.

(b) if  $\alpha > \gamma$ , then  $(s_k, \alpha)$  is larger than  $(s_l, \gamma)$ .

(2) Negation operator for linguistic 2-tuples:

$$\text{Neg}((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha)))$$

(3) The aggregation operators for linguistic 2-tuples were defined in [14], and their details are omitted herein.

### B. Linguistic preference relations and consistency measurements

Let  $A = \{A_1, A_2, \dots, A_n\}$  be a set of alternatives. When decision makers pairwise compare alternatives  $(A_i, A_j)$  using the linguistic term set  $S$ , they construct a linguistic preference

relation  $L = (l_{ij})_{n \times n}$ , where  $l_{ij}$  denotes the linguistic preference degree of  $A_i$  over  $A_j$ .

**Definition 2** [12], [13]. The linguistic preference relation  $L = (l_{ij})_{n \times n}$  is complete if  $l_{ij} \in S$  and  $l_{ij} = \text{Neg}(l_{ji})$  for all  $i, j = 1, 2, \dots, n$ .

Additive transitivity [1], [8], [15] is often used in decision making to measure the consistency of preference relations. The consistency measurement of linguistic preference relations based on the 2-tuple linguistic model is provided as below.

**Definition 3** [1]. Let  $L = (l_{ij})_{n \times n}$  be a complete linguistic preference relation based on  $S$ . The consistency index of  $L$  is defined as follows,

$$CI(L) = 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^n |\Delta^{-1}(l_{ij}) + \Delta^{-1}(l_{jk}) - \Delta^{-1}(l_{ik}) - g/2| \quad (3)$$

Clearly,  $CI(L) \in [0, 1]$ . A larger value of  $CI(L)$  indicates a better consistency of  $L$ .

In some cases decision makers cannot provide linguistic preferences for all the possible pairs of alternatives, which leads to the practical use of incomplete linguistic preference relations in decision making, a concept that is provided below.

**Definition 4** [21].  $L = (l_{ij})_{n \times n}$  is an incomplete linguistic preference relation when some of its elements are missing or unknown (not provided by the decision maker and denoted by the symbol “ $x$ ” herein), while the known elements (provided by the decision maker) satisfy  $l_{ij} \in S$  and  $l_{ij} = \text{Neg}(l_{ji})$ .

### C. The PIS-based numerical scale model

Dong et al. [6] defined the below concept of the numerical scale as an extension of the linguistic 2-tuples.

**Definition 5** [6]. Let  $S = \{s_0, s_1, \dots, s_g\}$  be a linguistic term set, and  $R$  be the set of real numbers. A function  $NS: S \rightarrow R$  is called a numerical scale of  $S$ , and  $NS(s_i)$  is the numerical index of  $s_i$ .

**Definition 6** [6]. Let  $S$  be defined as above. The numerical scale  $NS$  for  $(s_i, \alpha)$  is defined as follows:

$$NS(s_i, \alpha) = \begin{cases} NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)), & \alpha \geq 0 \\ NS(s_i) + \alpha \times (NS(s_i) - NS(s_{i-1})), & \alpha < 0 \end{cases} \quad (4)$$

If  $NS(s_i) < NS(s_{i+1})$ , for  $i = 0, 1, \dots, g-1$ , then the  $NS$  on  $S$  is ordered.

The inverse operator of the numerical scale  $NS$  of Definition 6 is [6]

$$NS^{-1}: R \rightarrow \bar{S} \quad (5)$$

with

$$NS^{-1}(r) = \begin{cases} \left( s_i, \frac{r - NS(s_i)}{NS^k(s_{i+1}) - NS^k(s_i)} \right), & NS(s_i) < r < \frac{NS(s_i) + NS(s_{i+1})}{2} \\ \left( s_i, \frac{r - NS(s_i)}{NS(s_i) - NS(s_{i-1})} \right), & \frac{NS(s_{i-1}) + NS(s_i)}{2} \leq r \leq NS(s_i) \end{cases}$$

The desired property of the above numerical scale to connect the 2-tuple linguistic models was discussed in [7].

The consistency measurement of complete linguistic preference relation based on  $NS$  is provided below.

**Definition 7** [6], [15].  $L = (l_{ij})_{n \times n}$  is a consistent complete linguistic preference relation with respect to the ordered

numerical scale  $NS$  if for all  $i, j, k = 1, 2, \dots, n$ :  $NS(l_{ij}) \in [0, 1]$  and  $NS(l_{ij}) + NS(l_{jk}) - NS(l_{ik}) = 0.5$ .

**Definition 8** [6]. Let  $L = (l_{ij})_{n \times n}$  be a complete linguistic preference relation and  $NS$  be the numerical scale on  $S$ . Then, the consistency index of  $L$  with respect to  $NS$  is

$$CI(L) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i < j < k} |NS(l_{ij}) + NS(l_{jk}) - NS(l_{ik}) - 0.5| \quad (6)$$

with  $NS(l_{ij}) \in [0, 1]$  for all  $i, j = 1, 2, \dots, n$ .

To handle the fact that words have multiple meanings for different people, Li et al. [18] presented a framework to handle linguistic information in the LGDM with PISs (Fig.1), which includes three processes: the individual semantics translation, numerical computation and individual semantics retractions.

The individual semantics translation is used to translate the linguistic terms into the PISs defined by the numerical values, in which the output activated the numerical computation to obtain a numerical value. The individual semantics retranslation is used to retranslate the output of numerical scales into linguistic values.

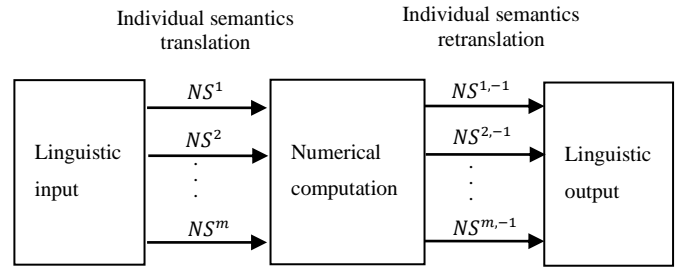


Fig.1. The framework for the linguistic model with PISs

In Fig. 1,  $NS^k$  is the ordered numerical scale on  $S$  of decision maker  $e_k$  ( $k = 1, 2, \dots, m$ ), and the value  $NS^k(s_i)$  represents the individual semantics of such decision maker  $e_k$  on the term  $s_i$  ( $i = 0, 1, \dots, g$ ). Furthermore, Li et al. [18-20] proposed consistency-driven optimization models to obtain the PNSs of linguistic terms for decision makers under different linguistic decision making contexts.

### III. CONSISTENCY-BASED PIS METHOD TO MANAGE INCOMPLETE LINGUISTIC PREFERENCE RELATIONS

In LGDM, the PISs reflect the different understanding of words among decision makers. To manage incomplete linguistic preference relations in a PIS context, in this section we propose a consistency-based methodology to estimate the unknown values of incomplete linguistic preference relations and to obtain the PNSs of linguistic expressions for decision makers.

#### A. The framework for the incomplete preference estimation method with PISs

Similar to the consistency measurement of a complete linguistic preference relation (see Definitions 3 and 8), we define the measure of consistency of an incomplete linguistic preference relation based on the additive transitivity. The condition that guarantees that the consistency of an incomplete linguistic preference relation  $L = (l_{ij})_{n \times n}$  can be measured is

the existence of at least three different preference values,  $l_{ij}, l_{jk}, l_{ik}$  ( $i \neq j \neq k$ ), that are known.

When studying incomplete linguistic preference relations, the following sets are needed,

$$V = \{(i, j) | l_{ij} \text{ is known}, (i, j) \in \{1, 2, \dots, n\}, i \neq j\}$$

$$ATV = \{(i, j, k) | (i, j), (j, k), (i, k) \in V\}$$

$V$  is the set of pairs of alternatives for which the decision makers provide linguistic preference values;  $ATV$  is the set of triplets of alternatives  $A_i, A_j, A_k$  for which the preference degrees over these alternatives are known. The consistency measurement for incomplete linguistic preference relations is provided as follows,

**Definition 9.** Let  $L = (l_{ij})_{n \times n}$  be an incomplete linguistic preference relation. Then the consistency index of  $L$  is

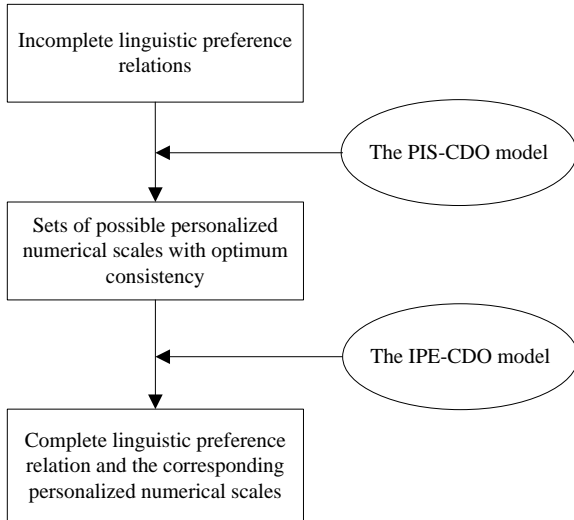
$$CI(L) = 1 - \frac{2}{3 * \#ATV} \sum_{i < j < k; (i, j, k) \in ATV} |NS(l_{ij}) + NS(l_{jk}) - NS(l_{ik}) - 0.5| \quad (7)$$

with  $NS(l_{ij}) \in [0, 1]$  for all  $i, j = 1, 2, \dots, n$ .

As mentioned before, being consistent implies that the decision makers are neither random nor illogical when expressing preferences. In order to estimate the missing values and also to guarantee the consistency of an incomplete linguistic preference relation, we propose a two-phase consistency-driven methodology:

(1) In the first phase, a PIS-CDO model is developed to optimize the consistency of the incomplete linguistic preference relation by generating the set of all possible PNSs for the known values.

(2) In the second phase, an IPE-CDO model to estimate the missing values of incomplete linguistic preference relation with PISs is established. This model guarantees the optimum consistency of both the original incomplete linguistic preference relation and its complete linguistic preference relation derived from the estimation process.



**Fig.2.** The framework for managing incomplete linguistic preference relations with PISs

Fig. 2 depicts the framework of the proposed methodology to manage incomplete linguistic preference relations.

### B. The PIS-CDO model with incomplete linguistic preference relations

This subsection provides a PIS-CDO model to obtain the set of all possible PNSs for an incomplete linguistic preference relation that guarantees it has the maximum consistency.

Let  $L^k$  be the incomplete linguistic preference relation provided by decision maker  $e_k$ . Let  $NS^k$  be the numerical scale associated with  $e_k$ . As per the previous definitions, let  $V^k$  and  $ATV^k$  be the sets of pairs and triplets of alternatives with known linguistic values given by decision maker  $e_k$ :

$$V^k = \{(i, j) | l_{ij}^k \text{ is known}, (i, j) \in \{1, 2, \dots, n\}, i \neq j\}$$

$$ATV^k = \{(i, j, z) | (i, j), (j, z), (i, z) \in V^k\}$$

As we aim to maximize the consistency of incomplete linguistic preference relations, the objective function of the PIS-CDO model will be:

$$\max CI(L^k) \quad (8)$$

where  $CI(L^k)$  is defined as per expression (7) of Definition 9.

The range of the numerical scale for each linguistic term associated with decision maker  $e_k$  is

$$NS^k(s_i) = \begin{cases} = 0, & i = 0 \\ \in \left[ \frac{i-0.5}{g}, \frac{i+0.5}{g} \right), & i = 1, 2, \dots, g-1; i \neq \frac{g}{2} \\ = 0.5, & i = \frac{g}{2} \\ = 1, & i = g \end{cases} \quad (9)$$

Thus, the constructed PIS-CDO model with incomplete linguistic preference relation  $L^k$  is

$$\begin{cases} \max CI(L^k) = 1 - \frac{2}{3 * \#ATV^k} \sum_{i < j < z; (i, j, z) \in ATV^k} |NS^k(l_{ij}^k) + NS^k(l_{jz}^k) - NS^k(l_{iz}^k) - 0.5| \\ \text{s.t.} \\ NS^k(s_0) = 0 \\ NS^k(s_q) \in \left( \frac{q-0.5}{g}, \frac{q+0.5}{g} \right], \quad q = 1, 2, \dots, g-1; q \neq g/2 \\ NS^k\left(s_{\frac{g}{2}}\right) = 0.5 \\ NS^k(s_g) = 1 \end{cases} \quad (10)$$

By solving Model (10), we generate the set of possible PNSs of linguistic terms for  $e_k$ , denoted  $APS = \{(NS^k(s_0), NS^k(s_1), \dots, NS^k(s_g)); \dots\}$ , that guarantee the maximum consistency of the provided incomplete linguistic preference relation  $L^k$ . The obtained optimum consistency index of  $L^k$  is denoted  $OCI(L^k)$  in this paper.

Model (10) provides a novel approach to measure the consistency of incomplete linguistic preference relations by setting the PNSs for linguistic terms. The obtained results will be used in the IPE-CDO model to make sure the optimum consistency of the incomplete linguistic preference relation will not be destroyed.

### C. The IPE-CDO model to estimate the missing values

This subsection proposes an IPE-CDO model to estimate the missing values of an incomplete linguistic preference relation and its personalized numerical meanings for the corresponding decision maker, while guaranteeing that the obtained complete linguistic preference relation has maximum consistency level.

Let  $L^k$  be defined as before. Let  $\bar{L}^k = (\bar{l}_{ij}^k)_{n \times n}$  be the complete linguistic preference relation associated to  $L^k$ . The objective of the optimization model with PISs that estimates the missing values will be

$$\max CI(\bar{L}^k) \quad (11)$$

where  $CI(\bar{L}^k)$  is defined as per expression (6) of Definition 8.

Let  $OCI(L^k)$  be the consistency index of the incomplete linguistic preference relation  $L^k$  obtained from Model (10). To guarantee the optimum consistency of  $L^k$  is not destroyed in the process of estimating missing values, the PNSs of the known linguistic values provided by decision maker  $e_k$  should belong to the set  $APS$ , and therefore

$$\{NS^k(\bar{l}_{ij}^k) | \bar{l}_{ij}^k \in V^k\} \subseteq APS \quad (12)$$

Eq. (12) can also be expressed as follows:

$$1 - \frac{2}{3 \times \#ATV^k} \sum_{i < j < z; (i,j,z) \in ATV^k} |NS^k(\bar{l}_{ij}^k) + NS^k(\bar{l}_{jz}^k) - NS^k(\bar{l}_{iz}^k) - 0.5| = OCI(L^k) \quad (13)$$

where  $\#ATV^k$  is the cardinality of set  $ATV^k$ .

The range for the numerical scales of linguistic terms for decision maker  $e_k$  is set as in Model (10), i.e.,

$$NS^k(s_q) \begin{cases} = 0, & q = 0 \\ \in [\frac{q-0.5}{g}, \frac{q+0.5}{g}], & q = 1, 2, \dots, g-1; i \neq \frac{g}{2} \\ = 0.5, & q = \frac{g}{2} \\ = 1, & q = g \end{cases} \quad (14)$$

In constructing the complete linguistic preference relation  $\bar{L}^k$ , its elements need to fulfill the following constraints

$$\begin{cases} \bar{l}_{ij}^k = l_{ij}^k, & l_{ij}^k \neq x \\ \bar{l}_{ij}^k \in S, & l_{ij}^k = x \end{cases} \quad (15)$$

Bringing all the above together results in the following IPE-CDO model for estimating the missing values of  $L^k$

$$\begin{cases} \max CI(\bar{L}^k) = 1 - \frac{4}{n(n-1)(n-2)} \\ \sum_{i < j < z} |NS^k(\bar{l}_{ij}^k) + NS^k(\bar{l}_{jz}^k) - NS^k(\bar{l}_{iz}^k) - 0.5| \\ s. t. \\ 1 - \frac{2}{3 \times \#ATV^k} \sum_{i < j < z; (i,j,z) \in ATV^k} |NS^k(\bar{l}_{ij}^k) + NS^k(\bar{l}_{jz}^k) - NS^k(\bar{l}_{iz}^k) - 0.5| = OCI(L^k) \\ NS^k(s_0) = 0 \\ NS^k(s_q) \in [\frac{q-0.5}{g}, \frac{q+0.5}{g}] \quad \text{for } q = 1, 2, \dots, g-1; q \neq g/2 \\ NS^k(s_{g/2}) = 0.5 \\ NS^k(s_g) = 1 \\ \bar{l}_{ij}^k = l_{ij}^k, & l_{ij}^k \neq x; i < j \\ \bar{l}_{ij}^k \in S, & l_{ij}^k = x; i < j \end{cases} \quad (16)$$

By solving Model (16), we estimate the missing values of  $L^k$  to generate its complete linguistic preference relation  $\bar{L}^k$  and further obtain the PNSs of linguistic terms for  $e_k$ ,  $NS^k(s_0), NS^k(s_1), \dots, NS^k(s_g)$ . Besides, we also obtain the

optimum consistency index of  $\bar{L}^k$ ,  $OCI(\bar{L}^k)$ . The obtained PNSs may be different for different decision makers, which agree with the actual different understanding of words by different decision makers.

From this two-phase method for managing incomplete linguistic preference relations, it is clear that the proposed methodology provides a novel consistency index of an incomplete linguistic preference relations by considering the decision makers' PISs in estimating the missing values when constructing its complete linguistic preference relation.

#### IV. NUMERICAL EXAMPLE

In this section, we present numerical examples to illustrate the proposed consistency-driven optimization models in managing incomplete linguistic preference relations.

Let  $S$  be the following linguistic term set

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$$

Let  $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$  and  $A = \{A_1, A_2, A_3, A_4, A_5\}$  be the set of decision makers and alternatives, respectively. We assume the following incomplete linguistic preference relations are provided by  $e_k$  ( $k = 1, 2, \dots, 8$ ),

$$\begin{aligned} L^1 &= \begin{pmatrix} - & s_4 & s_1 & x & x \\ s_2 & - & s_2 & s_3 & x \\ s_5 & s_4 & - & s_0 & x \\ x & s_3 & s_6 & - & s_2 \\ x & x & x & s_4 & - \end{pmatrix} \\ L^2 &= \begin{pmatrix} - & s_6 & x & x & s_0 \\ s_0 & - & x & s_3 & s_4 \\ x & x & - & x & x \\ x & s_3 & x & - & s_6 \\ s_6 & s_2 & x & s_0 & - \end{pmatrix} \\ L^3 &= \begin{pmatrix} - & x & x & x & x \\ x & - & s_1 & s_5 & s_0 \\ x & s_5 & - & s_6 & s_2 \\ x & s_1 & s_0 & - & x \\ x & s_6 & s_4 & x & - \end{pmatrix} \\ L^4 &= \begin{pmatrix} - & s_1 & x & s_4 & s_0 \\ s_5 & - & x & s_5 & x \\ x & x & - & x & x \\ s_2 & s_1 & x & - & s_6 \\ s_6 & x & x & s_0 & - \end{pmatrix} \\ L^5 &= \begin{pmatrix} - & x & s_2 & s_1 & x \\ x & - & s_5 & s_0 & x \\ s_4 & s_1 & - & s_4 & x \\ s_5 & s_6 & s_2 & - & x \\ x & x & x & x & - \end{pmatrix} \\ L^6 &= \begin{pmatrix} - & s_5 & s_1 & x & s_0 \\ s_1 & - & s_6 & s_2 & s_4 \\ s_5 & s_0 & - & s_3 & x \\ x & s_4 & s_3 & - & x \\ s_6 & s_2 & x & x & - \end{pmatrix} \end{aligned}$$

$$L^7 = \begin{pmatrix} - & x & x & x & s_1 \\ x & - & s_6 & s_5 & s_4 \\ x & s_0 & - & x & s_5 \\ x & s_1 & x & - & s_2 \\ s_5 & s_2 & s_1 & s_4 & - \end{pmatrix}$$

$$L^8 = \begin{pmatrix} - & x & s_5 & x & s_1 \\ x & - & x & s_4 & s_5 \\ s_1 & x & - & x & s_2 \\ x & s_2 & x & - & s_3 \\ s_5 & s_1 & s_4 & s_3 & - \end{pmatrix}$$

(1) Phase I: The PIS-CDO model with incomplete linguistic preference relations.

In this phase optimum consistency of  $L^k = (l_{ij}^k)_{5 \times 5}$  ( $k = 1, 2, \dots, 8$ ) is obtained via the generation of the PNSs for the linguistic terms. As an example, we illustrate this phase with  $L^1$  by building the consistency-driven optimization model (10) with following Eqs. (17)-(24).

$$\max CI(L^1) = 1 - \frac{1}{3}(|NS^1(s_4) + NS^1(s_2) - NS^1(s_1) - 0.5| + |NS^1(s_2) + NS^1(s_0) - NS^1(s_3) - 0.5|) \quad (17)$$

subject to

$$NS^1(s_0) = 0, \quad (18)$$

$$NS^1(s_1) \in (0.0833, 0.25], \quad (19)$$

$$NS^1(s_2) \in (0.25, 0.4167], \quad (20)$$

$$NS^1(s_4) \in (0.5833, 0.75], \quad (21)$$

$$NS^1(s_5) \in (0.75, 0.9167], \quad (22)$$

$$NS^1(s_3) = 0.5, \quad (23)$$

$$NS^1(s_6) = 1. \quad (24)$$

The solution of this model results in  $OCI(L^1) = 0.722$  with the set of PNSs,  $APS$ , that can include several possible PNSs, such as

$$\{NS^1(s_0) = 0; NS^1(s_1) = 0.25; NS^1(s_2) = 0.3331; NS^1(s_3) = 0.5; NS^1(s_4) = 0.5833; NS^1(s_5) \in (0.75, 0.9167]; NS^1(s_6) = 1\} \subseteq APS,$$

and

$$\{NS^1(s_0) = 0, NS^1(s_1) = 0.249, NS^1(s_2) = 0.375, NS^1(s_3) = 0.5, NS^1(s_4) = 0.583, NS^1(s_5) \in [0.75, 0.9167], NS^1(s_6) = 1\} \subseteq APS.$$

Similarly, we obtain the following optimum consistency indexes for  $L^k$  ( $k = 2, 3, \dots, 8$ ):  $OCI(L^2) = 0.5$ ,  $OCI(L^3) = 0.995$ ,  $OCI(L^4) = 0.6386$ ,  $OCI(L^5) = 0.687$ ,  $OCI(L^6) = 0.426$ ,  $OCI(L^7) = 0.829$  and  $OCI(L^8) = 0.91$ .

The obtained optimum consistency indexes  $OCI(L^k)$  ( $k = 1, 2, \dots, 8$ ) will be used as a constraint condition in Phase II to further guarantee the PISs of decision makers in solving the complete linguistic preference relation  $\bar{L}^k$ .

(2) Phase II: The IPE-CDO model to estimate the missing values.

In this phase, solving Model (16) results in the complete linguistic preference relation  $\bar{L}^k$ , i.e., the missing values of  $L^k$  ( $k = 1, 2, \dots, 8$ ) are estimated, and in the personalized numerical meanings for decision makers  $e_k$ , while ensuring the consistency of  $\bar{L}^k$  at the same time. Below, we illustrate this with the incomplete linguistic preference relation  $L^1$  (See Eqs. (25)-(36)).

$$\max CI(\bar{L}^1) = 1 - \frac{1}{15} \sum_{i < j < z} |NS^1(\bar{l}_{ij}^1) + NS^1(\bar{l}_{jz}^1) - NS^1(\bar{l}_{iz}^1) - 0.5| \quad (25)$$

subject to

$$1 - \frac{1}{3}(|NS^1(\bar{l}_{12}^1) + NS^1(\bar{l}_{23}^1) - NS^1(\bar{l}_{13}^1) - 0.5| + |NS^1(\bar{l}_{23}^1) + NS^1(\bar{l}_{34}^1) - NS^1(\bar{l}_{24}^1) - 0.5|) = 0.722, \quad (26)$$

$$NS^1(s_0) = 0, \quad (27)$$

$$NS^1(s_1) \in (0.0833, 0.25], \quad (28)$$

$$NS^1(s_2) \in (0.25, 0.4167], \quad (29)$$

$$NS^1(s_4) \in (0.5833, 0.75], \quad (30)$$

$$NS^1(s_5) \in (0.75, 0.9167], \quad (31)$$

$$NS^1(s_3) = 0.5, \quad (32)$$

$$NS^1(s_6) = 1, \quad (33)$$

$$\bar{l}_{12}^1 = s_4; \bar{l}_{13}^1 = s_1; \bar{l}_{23}^1 = s_2, \quad (34)$$

$$\bar{l}_{24}^1 = s_3; \bar{l}_{34}^1 = s_0; \bar{l}_{45}^1 = s_2, \quad (35)$$

$$\bar{l}_{14}^1 \in S; \bar{l}_{15}^1 \in S; \bar{l}_{25}^1 \in S; \bar{l}_{35}^1 \in S. \quad (36)$$

The solution of this model yields the following missing value estimations:  $\bar{l}_{14}^1 = s_2$ ,  $\bar{l}_{15}^1 = s_1$ ,  $\bar{l}_{25}^1 = s_1$ ,  $\bar{l}_{35}^1 = s_2$ . Thus, the complete linguistic preference relation  $\bar{L}^1$  is

$$\bar{L}^1 = \begin{pmatrix} - & s_4 & s_1 & s_2 & s_1 \\ s_2 & - & s_2 & s_3 & s_1 \\ s_5 & s_4 & - & s_0 & s_2 \\ s_4 & s_3 & s_6 & - & s_2 \\ s_5 & s_5 & s_4 & s_4 & - \end{pmatrix}$$

The PNSs for decision maker  $e_1$  are:  $NS^1(s_0) = 0$ ,  $NS^1(s_1) = 0.249$ ,  $NS^1(s_2) = 0.375$ ,  $NS^1(s_3) = 0.5$ ,  $NS^1(s_4) = 0.584$ , and  $NS^1(s_6) = 1$ .

Similarly, we obtain the rest of complete linguistic preference relations

$$\bar{L}^2 = \begin{pmatrix} - & s_6 & s_1 & s_0 & s_0 \\ s_0 & - & s_3 & s_3 & s_4 \\ s_5 & s_3 & - & s_1 & s_4 \\ s_6 & s_3 & s_5 & - & s_6 \\ s_6 & s_2 & s_2 & s_0 & - \end{pmatrix}$$

$$\bar{L}^3 = \begin{pmatrix} - & s_3 & s_1 & s_4 & s_0 \\ s_3 & - & s_1 & s_5 & s_0 \\ s_5 & s_5 & - & s_6 & s_2 \\ s_2 & s_1 & s_0 & - & s_0 \\ s_6 & s_6 & s_4 & s_6 & - \end{pmatrix}$$

$$\bar{L}^4 = \begin{pmatrix} - & s_1 & s_1 & s_4 & s_0 \\ s_5 & - & s_3 & s_5 & s_1 \\ s_5 & s_3 & - & s_5 & s_1 \\ s_2 & s_1 & s_1 & - & s_6 \\ s_6 & s_5 & s_5 & s_0 & - \end{pmatrix}$$

$$\bar{L}^5 = \begin{pmatrix} - & s_3 & s_2 & s_1 & s_0 \\ s_3 & - & s_5 & s_0 & s_0 \\ s_4 & s_1 & - & s_4 & s_1 \\ s_5 & s_6 & s_2 & - & s_2 \\ s_6 & s_6 & s_5 & s_4 & - \end{pmatrix}$$

$$\bar{L}^6 = \begin{pmatrix} - & s_5 & s_1 & s_1 & s_0 \\ s_1 & - & s_6 & s_2 & s_4 \\ s_5 & s_0 & - & s_3 & s_1 \\ s_5 & s_4 & s_3 & - & s_1 \\ s_6 & s_2 & s_5 & s_5 & - \end{pmatrix}$$

$$\bar{L}^7 = \begin{pmatrix} - & s_0 & s_1 & s_1 & s_1 \\ s_6 & - & s_6 & s_5 & s_4 \\ s_5 & s_0 & - & s_3 & s_5 \\ s_5 & s_1 & s_3 & - & s_2 \\ s_5 & s_2 & s_1 & s_4 & - \end{pmatrix}$$

$$\bar{L}^8 = \begin{pmatrix} - & s_0 & s_5 & s_1 & s_1 \\ s_6 & - & s_6 & s_4 & s_5 \\ s_1 & s_0 & - & s_1 & s_2 \\ s_5 & s_2 & s_5 & - & s_3 \\ s_5 & s_1 & s_4 & s_3 & - \end{pmatrix}$$

The  $OCI(\bar{L}^k)$  values ( $k = 1, 2, \dots, 8$ ) are given in Table I.

TABLE I  
OPTIMUM CONSISTENCY OF  $\bar{L}^k$  ( $k = 1, 2, \dots, 8$ )

	$\bar{L}^1$	$\bar{L}^2$	$\bar{L}^3$	$\bar{L}^4$
$OCI$	0.833	0.7	0.948	0.783
	$\bar{L}^5$	$\bar{L}^6$	$\bar{L}^7$	$\bar{L}^8$
$OCI$	0.831	0.76	0.878	0.945

The values of  $NS^k(s_i)$  for  $i = 0, 1, \dots, 6; k = 2, 3, \dots, 8$  are provided in Table II.

TABLE II  
PNSS OF LINGUISTIC TERMS FOR DECISION MAKERS  $e_k$  ( $k = 2, 3, \dots, 8$ )

	$NS^k(s_0)$	$NS^k(s_1)$	$NS^k(s_2)$	$NS^k(s_3)$	$NS^k(s_4)$	$NS^k(s_5)$	$NS^k(s_6)$
$k = 2$	0	0.25	—	0.5	0.75	—	1
$k = 3$	0	0.245	0.255	0.5	0.745	0.76	1
$k = 4$	0	0.25	—	0.5	0.584	0.834	1
$k = 5$	0	0.244	0.2562	0.5	0.584	0.76	1
$k = 6$	0	0.25	0.271	0.5	0.584	0.76	1
$k = 7$	0	0.25	0.4	0.5	0.75	0.76	1
$k = 8$	0	0.248	0.259	0.5	0.748	0.76	1

Fig.3 shows the difference among the PNSs of linguistic terms for  $e_k$  ( $k = 1, 2, \dots, 8$ ).

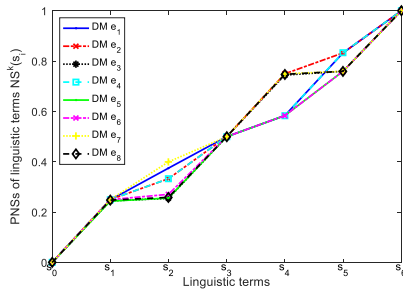


Fig.3. PNSs of linguistic terms for decision makers

From the numerical analysis we can see that the proposed methodology with PISs for managing incomplete linguistic preference relations results in different personalized numerical meanings for different decision makers and guarantees the optimum consistency of linguistic expressions, i.e. it reflects the different understandings of words by different decision makers.

## V. COMPARATIVE STUDY

In this section, we make a numerical analysis comparison with the existing method to manage incomplete linguistic preference relations that does not consider the decision makers' PISs.

Without personalizing individual semantics, the decision makers are assumed to have the same word semantics. In this case, instead of the numerical scale model, the 2-tuple linguistic model [14] is applied as the linguistic computational model for computing with words. In other words, the numerical scale function  $NS^k$  is replaced by the function  $\Delta^{-1}$  in the computation process, and the semantics of linguistic terms  $\{s_0, s_1, \dots, s_g\}$  for all the decision makers are  $\{0, 1, \dots, g\}$  because  $\Delta^{-1}(s_i) = i$ .

Let  $L^k$  be the incomplete linguistic preference relation provided by  $e_k$ , and  $V^k$  and  $ATV^k$  as previously defined. Next, we propose an optimization model to estimate missing values of  $L^k$  based on the 2-tuple linguistic model.

Let  $\bar{L}^k$  be the complete linguistic preference relation associated to  $L^k$ . Based on Definition 3, we measure the consistency of  $\bar{L}^k$  as follows,

$$CI(\bar{L}^k) = 1 - \frac{2}{3 \cdot \#ATV^k} \sum_{i < j < z; (i,j,z) \in ATV^k} |\Delta^{-1}(\bar{l}_{ij}^k) + \Delta^{-1}(\bar{l}_{jz}^k) - \Delta^{-1}(\bar{l}_{iz}^k) - 0.5| \quad (37)$$

To guarantee the obtained complete linguistic preference relation has maximum consistency index, the objective of the optimization model will be

$$\max CI(\bar{L}^k) \quad (38)$$

In constructing the complete linguistic preference relation  $\bar{L}^k$ , its elements need to fulfill the following conditions,

$$\begin{cases} \bar{l}_{ij}^k = l_{ij}^k, & l_{ij}^k \neq null, \\ \bar{l}_{ij}^k \in S, & l_{ij}^k = null \end{cases} \quad (39)$$

Therefore, the optimization model to estimate the missing values of  $L^k$  based on 2-tuple linguistic model is

$$\begin{cases} \max CI(\bar{L}^k) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i < j < z} |\Delta^{-1}(\bar{l}_{ij}^k) + \Delta^{-1}(\bar{l}_{jz}^k) - \Delta^{-1}(\bar{l}_{iz}^k) - 0.5| \\ s.t. \\ \bar{l}_{ij}^k = l_{ij}^k, & l_{ij}^k \neq null \\ \bar{l}_{ij}^k \in S, & l_{ij}^k = null \end{cases} \quad (40)$$

Solving the above model will result in the complete linguistic preference relations  $\bar{L}^k$  ( $k = 1, 2, \dots, m$ ) and the corresponding consistency index  $CI(\bar{L}^k)$ .

To illustrate the above model and to make a comparison with the proposed methodology with PISs, we provide numerical analysis using the set of eight incomplete preference relations  $L^k$  ( $k = 1, 2, \dots, 8$ ) of Section IV.

Based on the 2-tuple linguistic model, the following semantics of linguistic terms for decision makers are obtained:  $\Delta^{-1}(s_0) = 0$ ;  $\Delta^{-1}(s_1) = 0.167$ ;  $\Delta^{-1}(s_2) = 0.333$ ;  $\Delta^{-1}(s_3) = 0.5$ ;  $\Delta^{-1}(s_4) = 0.667$ ;  $\Delta^{-1}(s_5) = 0.833$  and  $\Delta^{-1}(s_6) = 1$ .

The optimization model to estimate the missing values of  $L^1$  is

$$\begin{aligned} \max CI(\bar{L}^1) = & 1 - \frac{4}{n(n-1)(n-2)} \sum_{i < j < z} |\Delta^{-1}(\bar{l}_{ij}^1) + \Delta^{-1}(\bar{l}_{jz}^1) \\ & - \Delta^{-1}(\bar{l}_{iz}^1) - 0.5| \end{aligned} \quad (41)$$

subject to

$$\bar{l}_{12}^1 = s_4; \bar{l}_{13}^1 = s_1; \bar{l}_{23}^1 = s_2, \quad (42)$$

$$\bar{l}_{24}^1 = s_3; \bar{l}_{34}^1 = s_0; \bar{l}_{45}^1 = s_2, \quad (43)$$

$$\bar{l}_{14}^1 \in S; \bar{l}_{15}^1 \in S; \bar{l}_{25}^1 \in S; \bar{l}_{35}^1 \in S. \quad (44)$$

By solving Eqs. (41)-(44), the following missing value estimations of  $L^1$  are obtained:  $\bar{l}_{14}^1 = s_4$ ,  $\bar{l}_{15}^1 = s_3$ ,  $\bar{l}_{25}^1 = s_2$  and  $\bar{l}_{35}^1 = s_3$ . Thus, the complete linguistic preference relation  $\bar{L}^1$  is

$$\bar{L}^1 = \begin{pmatrix} - & s_4 & s_1 & s_4 & s_3 \\ s_2 & - & s_2 & s_3 & s_2 \\ s_5 & s_4 & - & s_0 & s_3 \\ s_2 & s_3 & s_6 & - & s_2 \\ s_3 & s_4 & s_3 & s_4 & - \end{pmatrix}$$

Similarly, we estimate the missing values of  $L^k$  ( $k = 2, 3, \dots, 8$ ), which results in the following complete linguistic preference relations  $\bar{L}^k$  ( $k = 2, 3, \dots, 8$ ):

$$\begin{aligned} \bar{L}^2 &= \begin{pmatrix} - & s_6 & s_0 & s_0 & s_0 \\ s_0 & - & s_3 & s_3 & s_4 \\ s_6 & s_3 & - & s_3 & s_4 \\ s_6 & s_3 & s_3 & - & s_6 \\ s_6 & s_2 & s_2 & s_0 & - \end{pmatrix} \\ \bar{L}^3 &= \begin{pmatrix} - & s_5 & s_3 & s_6 & s_2 \\ s_1 & - & s_1 & s_5 & s_0 \\ s_3 & s_5 & - & s_6 & s_2 \\ s_0 & s_1 & s_0 & - & s_0 \\ s_4 & s_6 & s_4 & s_6 & - \end{pmatrix} \\ \bar{L}^4 &= \begin{pmatrix} - & s_1 & s_0 & s_4 & s_0 \\ s_5 & - & s_2 & s_5 & s_2 \\ s_6 & s_4 & - & s_6 & s_3 \\ s_2 & s_1 & s_0 & - & s_6 \\ s_6 & s_4 & s_3 & s_0 & - \end{pmatrix} \\ \bar{L}^5 &= \begin{pmatrix} - & s_0 & s_2 & s_1 & s_0 \\ s_6 & - & s_5 & s_0 & s_3 \\ s_4 & s_1 & - & s_4 & s_1 \\ s_5 & s_6 & s_2 & - & s_2 \\ s_6 & s_3 & s_5 & s_4 & - \end{pmatrix} \\ \bar{L}^6 &= \begin{pmatrix} - & s_5 & s_1 & s_1 & s_0 \\ s_1 & - & s_6 & s_2 & s_4 \\ s_5 & s_0 & - & s_3 & s_2 \\ s_5 & s_4 & s_3 & - & s_2 \\ s_6 & s_2 & s_4 & s_4 & - \end{pmatrix} \\ \bar{L}^7 &= \begin{pmatrix} - & s_0 & s_3 & s_2 & s_1 \\ s_6 & - & s_6 & s_5 & s_4 \\ s_3 & s_0 & - & s_2 & s_5 \\ s_4 & s_1 & s_4 & - & s_2 \\ s_5 & s_2 & s_1 & s_4 & - \end{pmatrix} \\ \bar{L}^8 &= \begin{pmatrix} - & s_0 & s_5 & s_1 & s_1 \\ s_6 & - & s_6 & s_4 & s_5 \\ s_1 & s_0 & - & s_1 & s_2 \\ s_5 & s_2 & s_5 & - & s_3 \\ s_5 & s_1 & s_4 & s_3 & - \end{pmatrix} \end{aligned}$$

The consistency index of the complete linguistic preference relations  $\bar{L}^k$  ( $k = 1, 2, \dots, 8$ ) are provided in Table III.

TABLE III  
CONSISTENCY INDEX OF  $\bar{L}^k$  ( $k = 1, 2, \dots, 8$ )

	$\bar{L}^1$	$\bar{L}^2$	$\bar{L}^3$	$\bar{L}^4$
<i>CI</i>	0.8	0.7	0.9334	0.767

	$\bar{L}^5$	$\bar{L}^6$	$\bar{L}^7$	$\bar{L}^8$
<i>CI</i>	0.8	0.733	0.866	0.889

From the comparison of the consistency indexes of Table III with the ones obtained from the proposed PIS method of Table I, it is obvious that the complete linguistic preference relations have lower consistency without considering PISs than considering PISs, i.e.  $CI(\bar{L}^k) \leq OCI(\bar{L}^k)$ . It can be observed in the case that PISs are not implemented, the semantics of the linguistic terms are fixed, which results the consistency of the complete linguistic preference relations is fixed as well. While in our proposed method, we obtain the PNSs of linguistic terms to guarantee the optimum consistency of the complete linguistic preference relations, which makes the higher consistency than that in the case without the consideration of PISs. The reason behind these results can be explained: not only the linguistic terms in a preference relation but also their semantics will strongly influence the consistency degree of a linguistic preference relation. In the literature of linguistic consistency, this issue has been ignored.

Therefore, it is shown that our methodology is more effective for managing incomplete linguistic preference relations because not only is able to model the different understandings of words by different decision makers, as expected in real cases, but also achieves completed linguistic preference relations with higher consistency index on the basis of consistency.

## VI. CONCLUSION

Words mean different things for different people, that is, decision makers have PISs regarding words. Incorporating the decision makers' PISs in LGDM results in a novel more realistic and effective methodology for managing incomplete linguistic preference relations. This paper proposes a consistency-driven two-phase methodology based on PISs to manage incomplete linguistic preference relations with two optimization models. Specifically, the first phase PIS-CDO model is developed to obtain optimum consistency of incomplete linguistic preference relation by establishing the sets of possible PNSs, which are subsequently integrated in the second phase IPE-CDO model to estimate the missing values of the incomplete linguistic preference relation based on PISs and the personalized numerical meanings for linguistic terms of the different decision makers.

The personalization of linguistic expressions and the estimation of missing values based on consistency measurement in the process of managing incomplete linguistic preference relations are evidence of the advantages of our proposed methodology. In this paper, the LGDM is based on the use of well-established but simple linguistic terms set and additive consistency. In future, we will investigate methods to manage incomplete linguistic information with ordinal consistency [43] and complex linguistic expressions, such as hesitant linguistic term sets [29], [33], [36], [39], linguistic distribution [38], [47], [50], flexible linguistic expression [37] and heterogeneous information [16].



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